Covariant derivative

- for submflds of $\mathbb{R}^n$
- pull-back bundles

Orientation, integration, Stokes thm.

Before break

- Finish covectors
- Abstract Riem mflds
- $\exists$ metric (Emb thm)
- Nash embedding

After break

AH, symmetric

Clarifications on v.b.

(1) Prop E v.b. $E' \subseteq E$ sub-bundle $\Rightarrow$ E/E' is a v.b.

note different meaning.

T $E \subseteq S \times \mathbb{R}^N$

can take $\tilde{E} := (E')^\perp \cap E$

then $E/E' \xrightarrow{\cong} \tilde{E}$,

$\longrightarrow$ cotang. bundle is v.b.

$\longrightarrow$ dual is v.b. $\left((\mathbb{R}^N)^* / E^*\right)$

Note $E_1 \otimes E_2 \subseteq \mathbb{R}^{N_1} \otimes \mathbb{R}^{N_2}$, so this is also a v.b.

Rmk 1 Just like for TS, T^*S is natural: a diffeo $\varphi : S \to \hat{S}$ induces an isomorphism $T^*S \to T^*\hat{S}$

$(p, \omega_p) \mapsto \left(d\varphi^{-1}_{\varphi(p)}\right)^* \omega_p$

Looks uglier, but actually nicer

(2) Pullback:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\ S \arrow[r, "f"'] & \hat{S} \end{tikzcd}$$

$$f^*E \subseteq S \times E \underset{\pi \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} \hat{S}$$

transverse

If $f: S \to \hat{S}$, get map of bundles $df: TS \longrightarrow f^*T\hat{S}$

$(p, V_p) \longleftrightarrow df_p V_p \in T_{f(p)}\hat{S}$

If $\hat{S} = \mathbb{R}$, $f^*T\hat{S} \cong \mathbb{R}$ canonically

get $df: TS \to \mathbb{R}$; df section of T^*S !

Pulling back is nice with sections

$df^*: f^*T^*\hat{S} \longrightarrow T^*S$ combined w/

$f^*: \sec(T^*\hat{S}) \longrightarrow \sec(f^*(T^*\hat{S}))$

gives map, which we call $f^*: \sec(T^*\hat{S}) \to \sec(T^*S)$

Chain rule: if $\varphi: S \to \hat{S}$, $f: \hat{S} \to \mathbb{R}$ $\quad d(f \circ \varphi) = df\, d\varphi$

$\Rightarrow \varphi^* df = d(f \circ \varphi)$

Rmk If $(x^1 \dots x^d)$ loc. coords on S, then $dx^1 \dots dx^n$ trivialize T^*S ($\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^d}$) triv. TS

sometimes written I, or g.

If $S \subseteq \mathbb{R}^n$, the first fund form is a section of $\bigotimes^2 T^*S$ which is symmetric and positive definite

The Rmk ! justifies conflating this w/ its coord expression.

Defn A Riemannian metric on a manifold is a sym, pos def section of $\bigotimes^2 T^*S$. Usually called g. Not connected to the embedding!

But prop Every manifold has a Riem metric (choose emb, take first F.F.)

Thm (Nash) Every Riem. mfld has an embedding to $\mathbb{R}^n$ which is isometric (i.e. s.t. $g = I$)

Abstract Riemannian manifolds:

<u>Recall</u> If $S_0 := \coprod_{\text{countably many } \alpha} \mathcal{U}_\alpha^{\subseteq \mathbb{R}^d}$, $R \subseteq S_0 \times S_0$ dim d { closed, π. sub.

$\longrightarrow$ S_0/R is an abstract mfd of dim d.

(resp. T^*)

In such case a section of $T(S_0/R)$ is the same as a section of T^*S_0 whose two pull-backs to R agree (c.f. exercises) (or a single ∞-dim'l v.b.)

<u>Tensor algebra</u> A collection of <u>natural</u> v.b.'s on S, i.e. s.t. a diffeo $S \to \tilde{S}$ induces an isom $E(S) \to E(\tilde{S})$.

Multilinear forms on V $\sim \bigotimes^r V^*$

$$\Phi : \underbrace{V \times \dots \times V}_{r} \to \mathbb{R}$$

If $V = T_p\mathbb{R}^n$ ($= \mathbb{R}^n$) coords $\frac{\partial}{\partial x^i}$,

then a basis for $\bigotimes^r V^*$ is $dx_{i_1} \otimes \dots \otimes dx_{i_r}$

The tensor algebra of V is $\leftarrow$ concatenation : not commutative!

$$\bigotimes^\bullet V^* = \bigoplus_r \bigotimes^r V$$

a generalization:

Let
$$\bigotimes^\bullet (V \oplus V^*) = \bigoplus_{r \in \mathbb{N}} \bigotimes^r (V \oplus V^*)$$
$$= \bigoplus_{p,q \in \mathbb{N}} \bigotimes^p V \otimes \bigotimes^q V^*$$

If $V = \mathbb{R}^n$, an elt is eg. $dx^i \otimes \frac{\partial}{\partial x^j} \otimes dx^k$ (canon isom to all dx^i's first, if you like

<u>Note</u> $V \otimes V^* \cong \mathrm{End}(V)$

$$\frac{\partial}{\partial x_i} \otimes dx^j \longmapsto \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & \\ & 1 \end{bmatrix} \end{array}$$

Contraction: $V \otimes V^* \longrightarrow \mathbb{R}$ is the trace.

$\frac{\partial}{\partial x^i} \otimes dx^j \mapsto \delta_i^j$

extends to tensor algebra.

By gen'n, $\bigotimes^r (TS \otimes T^*S)$ is a v.b. on S, and a natural one.

sections are tensor fields (or sometimes just tensors).

mult & contr. are natural.

Einstein notation

$\sum_{i,j,k} T_i{}^j{}_k \, dx^i \otimes \frac{\partial}{\partial x^j} \otimes dx^k$ gen'l elt

written $T_i{}^j{}_k$

- tensoring convention
- contraction convention

Extra confusion: now g_{ij} is also shorthand for the (coord.-invariant) metric g.